

# ON A THEOREM BY BOJANOV AND NAIDENOV APPLIED TO FAMILIES OF GEGENBAUER-SOBOLEV POLYNOMIALS

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ABSTRACT. Let  $\{Q_{n,\lambda}^{(\alpha)}\}_{n \geq 0}$  be the sequence of monic orthogonal polynomials with respect to the Gegenbauer-Sobolev inner product

$$\langle f, g \rangle_S := \int_{-1}^1 f(x)g(x)(1-x^2)^{\alpha-\frac{1}{2}} dx + \lambda \int_{-1}^1 f'(x)g'(x)(1-x^2)^{\alpha-\frac{1}{2}} dx,$$

where  $\alpha > -\frac{1}{2}$  and  $\lambda \geq 0$ . In this paper we use a recent result due to B.D. Bojanov and N. Naidenov [3], in order to study the maximization of a local extremum of the  $k$ th derivative  $\frac{d^k}{dx^k} Q_{n,\lambda}^{(\alpha)}$  in  $[-M_{n,\lambda}, M_{n,\lambda}]$ , where  $M_{n,\lambda}$  is a suitable value such that all zeros of the polynomial  $Q_{n,\lambda}^{(\alpha)}$  are contained in  $[-M_{n,\lambda}, M_{n,\lambda}]$  and the function  $|Q_{n,\lambda}^{(\alpha)}|$  attains its maximal value at the end-points of such interval. Also, some illustrative numerical examples are presented.

*Key words and phrases:* Sobolev orthogonal polynomials; oscillating polynomials.

## 1. INTRODUCTION

Extremal properties for general orthogonal polynomials is an interesting subject in approximation theory and their applications permeate many fields in science and engineering [5, 18, 21, 28, 29]. Although it may seem an old subject from the view point of the standard orthogonality [5, 18, 29], this is not the case neither in the general setting (cf. [11–14, 20]) nor from the view point of Sobolev orthogonality, where it remains like a partially explored subject [1]. In fact, new results continue to appear in some recent publications [10–12, 24, 26, 27].

Let  $d\mu(x) = (1-x^2)^{\alpha-\frac{1}{2}} dx$  with  $\alpha > -\frac{1}{2}$ , be the Gegenbauer measure supported on the interval  $[-1, 1]$ . We consider the following Sobolev inner product on the linear space  $\mathbb{P}$  of polynomials with real coefficients.

$$(1.1) \quad \langle f, g \rangle_S := \int_{-1}^1 f(x)g(x)d\mu(x) + \lambda \int_{-1}^1 f'(x)g'(x)d\mu(x),$$

where  $\lambda \geq 0$ . Let  $\{Q_{n,\lambda}^{(\alpha)}\}_{n \geq 0}$  denote the sequence of monic orthogonal polynomials with respect to (1.1). These polynomials are usually called monic Gegenbauer-Sobolev polynomials [7, 8, 15–17, 25] and it is known that the zeros of these polynomials are in the real line [15, 16], and

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therefore they belong to other important class of algebraic polynomials, namely the oscillating polynomials [3, 19].

The main result of [3] allows to guarantee the maximal absolute value of higher derivatives of a symmetric oscillating polynomial on a finite interval are attained in the end-points of such interval, whenever the maximal absolute value of the polynomial is attained in the end-points of that interval. Then, [3, Section 4] contains a brief explanation about applications of previous result to orthogonal polynomials on the real line associated to symmetric weights. We focus our attention on that last part of [3, Section 4] in order to enlarge the range of application of [3, Theorem 1] to the class of Gegenbauer-Sobolev polynomials corresponding to the inner product (1.1).

The paper is structured as follows. Section 2 provides some background about structural properties of the Gegenbauer and Gegenbauer-Sobolev polynomials corresponding to the inner product (1.1), respectively. Section 3 contains some well-known characteristics of the class of oscillating polynomials on a finite interval. We also state there our main result (see Theorem 3.2) and provide some illustrative numerical examples. Throughout this paper, the notation  $u_n \cong v_n$  means that the sequence  $\left\{\frac{u_n}{v_n}\right\}_n$  converges to 1 as  $n \rightarrow \infty$ . We will denote by  $\mathbb{P}_n$  and  $\|f\|_\infty$ , the space of polynomials of degree at most  $n$  and the uniform norm of  $f$  on the interval in consideration, respectively. Any other standard notation will be properly introduced whenever needed.

## 2. BASIC FACTS: GEGENBAUER AND GEGENBAUER-SOBOLEV ORTHOGONAL POLYNOMIALS

For  $\alpha > -\frac{1}{2}$  we denote by  $\{\hat{C}_n^{(\alpha)}\}_{n \geq 0}$  the sequence of Gegenbauer polynomials, orthogonal on  $[-1, 1]$  with respect to the measure  $d\mu(x)$  (cf. [29, Chapter IV]), normalized by

$$\hat{C}_n^{(\alpha)}(1) = \frac{\Gamma(n+2\alpha)}{n!\Gamma(2\alpha)}.$$

It is clear that this normalization does not allow  $\alpha$  to be zero or a negative integer. Nevertheless, the following limits exist for every  $x \in [-1, 1]$  (see [29, formula (4.7.8)].)

$$\lim_{\alpha \rightarrow 0} \hat{C}_0^{(\alpha)}(x) = T_0(x), \quad \lim_{\alpha \rightarrow 0} \frac{\hat{C}_n^{(\alpha)}(x)}{\alpha} = \frac{2}{n} T_n(x),$$

where  $T_n$  is the  $n$ th Chebyshev polynomial of the first kind. In order to avoid confusing notation, we define the sequence  $\{\hat{C}_n^{(0)}\}_{n \geq 0}$  as follows.

$$\hat{C}_0^{(0)}(1) = 1, \quad \hat{C}_n^{(0)}(1) = \frac{2}{n}, \quad \hat{C}_n^{(0)}(x) = \frac{2}{n} T_n(x), \quad n \geq 1.$$

We denote the  $n$ th monic Gegenbauer orthogonal polynomial by

$$(2.2) \quad C_n^{(\alpha)}(x) = (h_n^\alpha)^{-1} \hat{C}_n^{(\alpha)}(x),$$

where the constant  $h_n^\alpha$  (cf. [29, formula (4.7.31)]) is given by

$$(2.3) \quad h_n^\alpha = \frac{2^n \Gamma(n+\alpha)}{n! \Gamma(\alpha)}, \quad \alpha \neq 0,$$

$$(2.4) \quad h_n^0 = \lim_{\alpha \rightarrow 0} \frac{h_n^\alpha}{\alpha} = \frac{2^n}{n}, \quad n \geq 1.$$

Then for  $n \geq 1$ , we have  $C_n^{(0)}(x) = \lim_{\alpha \rightarrow 0} (h_n^\alpha)^{-1} \hat{C}_n^{(\alpha)}(x) = \frac{1}{2^{n-1}} T_n(x)$ .

**PROPOSITION 2.1.** *Let  $\{C_n^{(\alpha)}\}_{n \geq 0}$  be the sequence of monic Gegenbauer orthogonal polynomials. Then the following statements hold.*

(1) *Three-term recurrence relation. For every  $n \in \mathbb{N}$ ,*

$$(2.5) \quad x C_n^{(\alpha)}(x) = C_{n+1}^{(\alpha)}(x) + \gamma_n^{(\alpha)} C_{n-1}^{(\alpha)}(x), \quad \alpha > -\frac{1}{2}, \quad \alpha \neq 0,$$

*with initial conditions  $C_0^{(\alpha)}(x) = 1$ ,  $C_1^{(\alpha)}(x) = x$ , and recurrence coefficient  $\gamma_n^{(\alpha)} = \frac{n(n+2\alpha-1)}{4(n+\alpha)(n+\alpha-1)}$ .*

(2) *For every  $n \in \mathbb{N}$  (see [29, formula (4.7.15)]),*

$$(2.6) \quad \|C_n^{(\alpha)}\|_\mu^2 = \int_{-1}^1 [C_n^{(\alpha)}(x)]^2 d\mu(x) = \pi 2^{1-2\alpha-2n} \frac{n! \Gamma(n+2\alpha)}{\Gamma(n+\alpha+1) \Gamma(n+\alpha)}.$$

(3) *Structure relation (see [29, formula (4.7.29)]). For every  $n \geq 2$ ,*

$$(2.7) \quad C_n^{(\alpha-1)}(x) = C_n^{(\alpha)}(x) + \xi_{n-2}^{(\alpha)} C_{n-2}^{(\alpha)}(x),$$

*where*

$$(2.8) \quad \xi_n^{(\alpha)} = \frac{(n+2)(n+1)}{4(n+\alpha+1)(n+\alpha)}, \quad n \geq 0.$$

(4) *For every  $n \in \mathbb{N}$  (see [29, formula (4.7.14)]),*

$$(2.9) \quad \frac{d}{dx} C_n^{(\alpha)}(x) = n C_{n-1}^{(\alpha+1)}(x).$$

Some well-known properties of the monic Gegenbauer-Sobolev orthogonal polynomials corresponding to the inner product (1.1) are the following.

**PROPOSITION 2.2.** *Let  $\{Q_{n,\lambda}^{(\alpha)}\}_{n \geq 0}$  be the sequence of monic orthogonal polynomials with respect to (1.1). Then the following statements hold.*

(1) *The polynomials  $Q_{n,\lambda}^{(\alpha)}$  are symmetric, i.e.,*

$$(2.10) \quad Q_{n,\lambda}^{(\alpha)}(-x) = (-1)^n Q_{n,\lambda}^{(\alpha)}(x).$$

(2) *The zeros of  $Q_{n,\lambda}^{(\alpha)}$  are real and simple, and they interlace with the zeros of the monic Gegenbauer orthogonal polynomials  $C_n^{(\alpha)}$ . Furthermore, for  $\alpha \geq \frac{1}{2}$  they are all contained in the interval  $[-1, 1]$  and for  $-\frac{1}{2} < \alpha < \frac{1}{2}$  there is at most a pair of zeros symmetric with respect to the origin outside the interval  $[-1, 1]$ , (cf. [15, 16]).*

(3) *[15, Lemma 5.1]. For  $\alpha \geq \frac{1}{2}$ , we have  $Q_{n,\lambda}^{(\alpha)}(1) > 0$ .*

It is worthwhile to point out that in the case  $-\frac{1}{2} < \alpha < \frac{1}{2}$ , no global properties about the sign  $Q_{n,\lambda}^{(\alpha)}(1)$  can be deduced (cf. [15].)

However, the location of zeros of Sobolev orthogonal polynomials is not a trivial problem. For instance, if we consider  $(\mu_0, \mu_1)$  a vector of compactly supported positive measures on the real line with finite total mass and the following Sobolev inner product on the linear space  $\mathbb{P}$  of polynomials with real coefficients.

$$(2.11) \quad \langle f, g \rangle_{(\mu_0, \mu_1)} := \int f(x)g(x)d\mu_0(x) + \int f'(x)g'(x)d\mu_1(x),$$

then, simple examples show that the zeros of these Sobolev orthogonal polynomials do not necessarily remain in the convex hull of the union of the supports of the measures  $\mu_k$ ,  $k = 0, 1$ , and they can be complex. In this regard some numerical experiments may be found in [9]. In particular, the boundedness of the zeros of Sobolev orthogonal polynomials is an open problem [1, 16], but as was stated in [10], it could be obtained as a consequence of the boundedness of the multiplication operator  $Mf(z) = zf(z)$ : If  $M$  is bounded and  $\|M\|$  is its operator norm (induced by (2.11)), then all the zeros of the Sobolev orthogonal polynomials  $Q_n$  are contained in the disc  $\{z \in \mathbb{C} : |z| \leq \|M\|\}$ .

Indeed, if  $x_0$  is a zero of  $Q_n$  then  $xp(x) = x_0p(x) + Q_n(x)$  for a polynomial  $p \in \mathbb{P}_{n-1}$ . Since  $p$  and  $Q_n$  are orthogonal, we get

$$|x_0|^2 \|p\|_{(\mu_0, \mu_1)}^2 = \|xp\|_{(\mu_0, \mu_1)}^2 - \|Q_n\|_{(\mu_0, \mu_1)}^2 \leq \|xp\|_{(\mu_0, \mu_1)}^2 = \|Mp\|_{(\mu_0, \mu_1)}^2 \leq \|M\|^2 \|p\|_{(\mu_0, \mu_1)}^2,$$

which yields the above result.

Thus, in the last decades the question whether or not the multiplication operator  $M$  is bounded has been a topic of interest to investigators in the field, since it turns out to be a key for the location of zeros and for establishing the asymptotic behavior of orthogonal polynomials with respect to diagonal (or non-diagonal) Sobolev inner products (cf. [16, 26, 27] and the references therein)

From the structure relation (2.7) and [17, formula (3)] (see also [7, Proposition 1]) the following connection formula can be obtained.

PROPOSITION 2.3. *For  $\alpha > -\frac{1}{2}$ ,*

$$(2.12) \quad C_n^{(\alpha-1)}(x) = Q_{n,\lambda}^{(\alpha)}(x) - d_{n-2}(\alpha)Q_{n-2,\lambda}^{(\alpha)}(x), \quad n \geq 2,$$

where

$$(2.13) \quad d_n(\alpha) = \xi_n^{(\alpha)} \frac{\|C_n^{(\alpha)}\|_{\mu}^2}{\|Q_{n,\lambda}^{(\alpha)}\|_S^2}.$$

Moreover,

$$(2.14) \quad d_n(\alpha) \cong \frac{1}{16\lambda n^2}.$$

### 3. MAXIMIZATION OF LOCAL EXTREMUM OF THE DERIVATIVES FOR FAMILIES OF GEGENBAUER-SOBOLEV POLYNOMIALS

A polynomial  $P \in \mathbb{P}$  is said oscillating (see [2–4, 19, 22, 23]) if it has all its zeros on the real line  $\mathbb{R}$ . For example, the classical orthogonal polynomials on subsets of  $\mathbb{R}$  (Hermite, Laguerre and Jacobi polynomials [6, 20, 29]), orthogonal polynomials for weights in the Nevai class  $M(0, 1)$  [21], including those orthogonal with respect to weights belonging to Levin-Lubinsky class  $\mathcal{W}$  [13], and a broad class of Sobolev orthogonal polynomials [7, 9, 15–17, 25] constitute an important family of oscillating polynomials. Usually, when all zeros of a polynomial  $P \in \mathbb{P}_n$  with  $\deg(P) = n$ , are contained in a given finite interval  $[a, b]$ , it is called oscillating polynomial on  $[a, b]$ , (see [3, 19].)

We denote by  $\text{Osc}(\mathbb{R})$  and  $\text{Osc}[a, b]$  the classes of oscillating polynomials on  $\mathbb{R}$  and  $[a, b]$ , respectively. For any  $P \in \text{Osc}[a, b]$  with  $\deg(P) = n$ , we consider the vector  $\mathbf{h}(P) = (h_0(P), \dots, h_n(P))$ , where  $h_j(P) = |P(t_j)|$ ,  $0 \leq j \leq n$ ,  $t_0 = a$ ,  $t_n = b$ , and  $t_1 \leq t_2 \leq \dots \leq t_{n-1}$  are the zeros of  $P'$ .

Amongst the main characteristics of the class  $\text{Osc}[a, b]$  we list the following.

- i)  $P' \in \text{Osc}[a, b]$ , for all  $P \in \text{Osc}[a, b]$ .
- ii) Any  $P \in \text{Osc}[a, b]$  is completely determined by its local extrema and the values at the end-points of the interval  $[a, b]$  (cf. [2, Theorem 2], [4, Remark 1].)
- iii) For  $P \in \text{Osc}[a, b]$  with  $\deg(P) = n$ , there exists a monotone dependence of the parameters  $h_j(P')$  on the parameters  $h_0(P), \dots, h_n(P)$  of  $P$  (cf. [4, Lemma 3].)
- iv) If  $P \in \text{Osc}[a, b]$  with  $\deg(P) \geq 3$  and  $\|P\| = |P(a)|$ , then each local extremum of  $P'$  from the first half (i.e., with an index less than or equal to  $\lfloor \frac{n-1}{2} \rfloor$ , and  $\lfloor t \rfloor$  denoting the integer part of  $t$ ) is smaller in absolute value than  $|P'(a)|$ .

More precisely, the property iv) was stated in the following theorem.

**THEOREM 3.1.** ([3, Theorem 1]) *Let  $P \in \text{Osc}[a, b]$  with  $\deg(P) \geq 3$ . Assume that  $\|P\|_\infty = |P(a)| = 1$ . Then*

$$(3.15) \quad |P'(\tau_j)| < |P'(a)|, \text{ for } j = 0, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor,$$

where  $\tau_1 \leq \dots \leq \tau_{n-2}$  are the zeros of  $P''$ .

**COROLLARY 3.1.** ([3, Corollary 1]) *Let  $P \in \text{Osc}[-1, 1]$  be a symmetric polynomial, with  $\deg(P) = n$ . Assume that  $\|P\|_\infty = |P(1)| = 1$ . Then,*

$$(3.16) \quad \|P^{(k)}\|_\infty = |P^{(k)}(1)|, \text{ for } k = 1, \dots, n.$$

As a consequence of the combination of Theorem 3.1 (or Corollary 3.1) and the structural properties of the sequence  $\{Q_{n,\lambda}^{(\alpha)}\}_{n \geq 0}$  given in the previous section, we can obtain the maximization of local extremum of the derivatives for the sequence  $\{Q_{n,\lambda}^{(\alpha)}\}_{n \geq 0}$  as follows.

Let  $\{Q_{n,\lambda}^{(\alpha)}\}_{n \geq 0}$  be the sequence of monic orthogonal polynomials with respect to (1.1). Let us consider  $x_{n,\lambda}^{\alpha,[1]} < x_{n,\lambda}^{\alpha,[2]} < \dots < x_{n,\lambda}^{\alpha,[n]}$  the zeros of the Gegenbauer-Sobolev polynomial  $Q_{n,\lambda}^{(\alpha)}$  and  $N$  the maximum value attained by  $|Q_{n,\lambda}^{(\alpha)}(x)|$  on the interval  $[x_{n,\lambda}^{\alpha,[1]}, x_{n,\lambda}^{\alpha,[n]}]$ . Then  $M_{n,\lambda}$  can be defined to be the minimal real point such that  $x_{n,\lambda}^{\alpha,[n]} < M_{n,\lambda}$  and  $|Q_{n,\lambda}^{(\alpha)}(M_{n,\lambda})| = N$ , i.e.,  $M_{n,\lambda}$  is the point closest to  $x_{n,\lambda}^{\alpha,[n]}$  where the maximal absolute value of the polynomial  $Q_{n,\lambda}^{(\alpha)}$  is attained. Notice that  $M_{n,\lambda}$  also depends on the parameter  $\alpha$  and  $Q_{n,\lambda}^{(\alpha)} \in \text{Osc}[-M_{n,\lambda}, M_{n,\lambda}]$ . Thus, we can consider the following normalized polynomials

$$(3.17) \quad q_{n,\lambda}^{(\alpha)}(x) := \frac{Q_{n,\lambda}^{(\alpha)}(x)}{Q_{n,\lambda}^{(\alpha)}(M_{n,\lambda})}, \quad x \in [-M_{n,\lambda}, M_{n,\lambda}], \quad n \geq 0.$$

**THEOREM 3.2.** *Let  $\{q_{n,\lambda}^{(\alpha)}\}_{n \geq 0}$  be the sequence of orthogonal polynomials given in (3.17). Then  $\left| \frac{d^k}{dx^k} q_{n,\lambda}^{(\alpha)} \right|$  attains its maximal value on the interval  $[-M_{n,\lambda}, M_{n,\lambda}]$  at the end-points, for  $\alpha > -\frac{1}{2}$  and  $1 \leq k \leq n$ .*

*Proof.* It suffices to follow the proof of Theorem 3.1 (or Corollary 3.1) given in [3, Theorem 1 (or Corollary 1)] by making the corresponding modifications.  $\square$

Notice that from a numerical point of view the value  $M_{n,\lambda}$  can be difficult to obtain for  $n$  large enough. However, for any value  $K > 0$  such that  $N < |Q_{n,\lambda}^{(\alpha)}(x)|$  for  $x < -K$  and  $x > K$ , the result of Theorem 3.2 remains true on the interval  $[-K, K]$ .

We finish this section providing some illustrative numerical examples (with the help of MAPLE) about the above result for different values of  $n$ ,  $\alpha$  and  $\lambda$  (see Figure 1 and Figure 2 below).

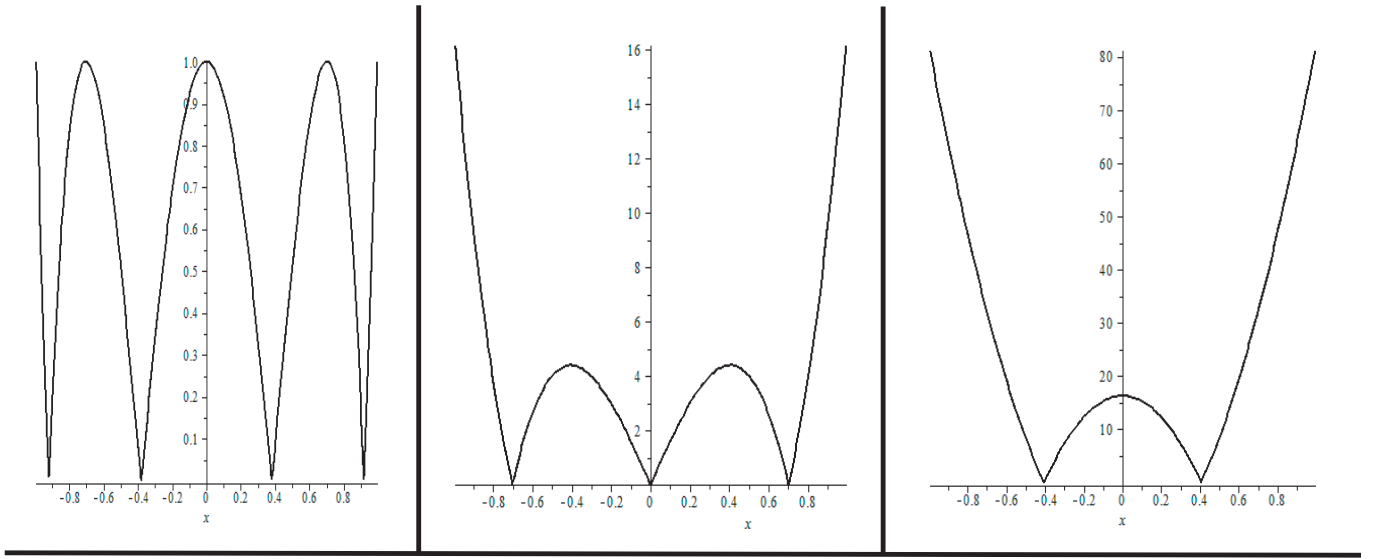


FIGURE 1. Graphics of  $|\frac{d^k}{dx^k} q_{n,\lambda}^{(\alpha)}|$  for  $n = 4$ ,  $\alpha = \lambda = 1$ ,  $M_{n,\lambda} = 0.9926198253$  and  $k = 0, 1, 2$ , respectively.

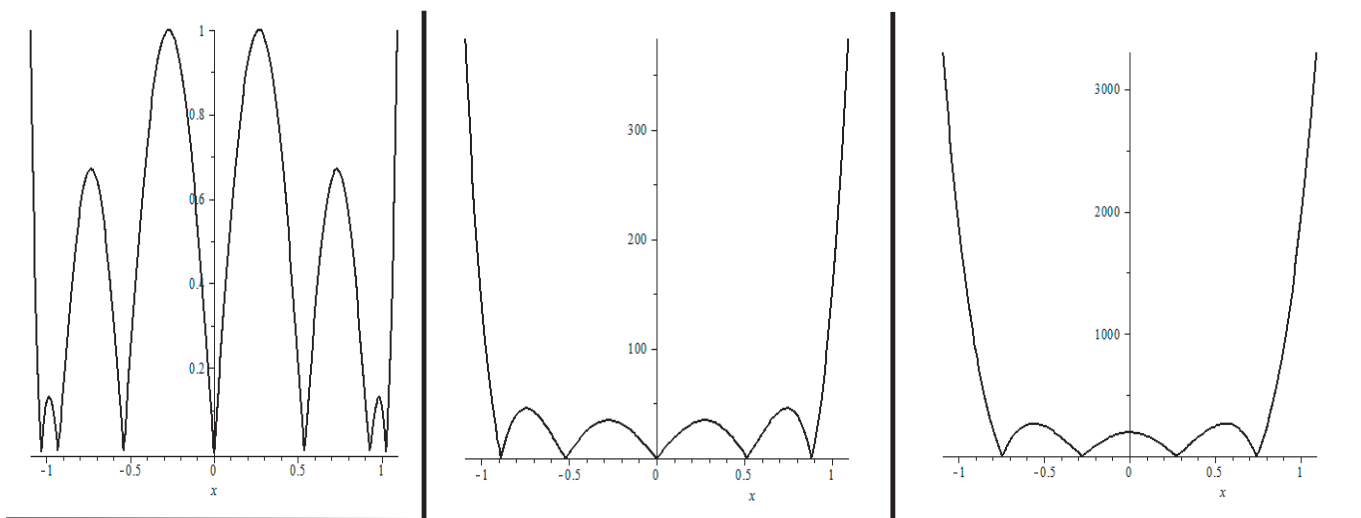


FIGURE 2. Graphics of  $|\frac{d^k}{dx^k} q_{n,\lambda}^{(\alpha)}|$  for  $n = 7$ ,  $\alpha = -\frac{1}{4}$ ,  $\lambda = \frac{1}{2}$ ,  $M_{n,\lambda} = 1.091516326$  and  $k = 0, 2, 3$ , respectively.

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